Review of Tetration Properties and an Alternative Theory on Tetration with Non-Integer Heights by F. Augustine

Introduction to Tetration

Addition, multiplication, and exponentiation can be called, collectively, hyperoperations.

Addition, multiplication, and exponentiation can be called the first, second, and third hyperoperations, respectively, while tetration is the fourth hyperoperation.

Just as multiplication is repeated addition and exponentiation is repeated multiplication, tetration is repeated exponentiation.

The typical notation for tetration is ${}^{3}2$, where ${}^{3}2 = 2^{(2^2)} = 16$, where 2 is referred to as the base, and 3 is referred to as the height.

Each hyperoperation has one or two inverse operations. Because they are commutative, addition and multiplication have one inverse function, subtraction and division, respectively. Exponentiation is not commutative, and therefore has two inverse functions, roots and logarithms.

Tetration is also not commutative and therefore has two inverse functions. No standard notation or nomenclature has been developed for these. Therefore in this paper we will use the following notation:

T-root, as in the 3rd t-root 16, $\overline{16}\sqrt{}^{3} = 2$. If the 3 is omitted, like the root function, it is assumed to be the 2nd t-root.

T-log, as in the t-log base 2 of 16, $\text{tlog}_2 16 = 3$.

Based on the definition of tetration, ${}^{n}x = x^{({}^{n-1}x)}$. In addition, $\log_x {}^{n}x = {}^{n-1}x$. Therefore, since $x = {}^{1}x$, ${}^{0}x = 1$, and ${}^{-1}x = 0$.

Properties of ${}^{n}x$

The properties of ${}^{n}x$, when n is an integer, are different for odd values of n versus even values, as shown in Figure 1.

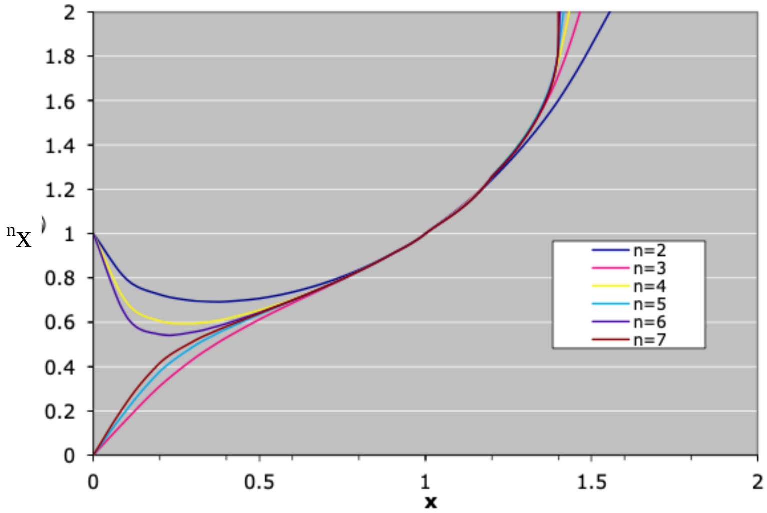


Figure 1

When n is even, there is a minimum value greater than 0 and less than 1. For ^2x, the coordinates of the minimum can be derived to be x=1/e, y= $e^{-1/e}$. When n is odd, the minimum is at x=0.

For positive real values of x, the behavior of $^nx = y$ can be very different depending on the value of x.

For example, if x is equal to or greater than 2, the value of ^nx goes up very rapidly with n. Examples:

$$^22 = 4,\ ^32 = 16,\ ^42 = 65536,\ ^52 > 10^{100}$$

$$^23 = 27,\ ^33 = 7.625 \times 10^{12},\ ^43 > 10^{100}$$

For values of a slightly greater than 1, the value of $^nx = y$ slowly increases with n eventually asymptotically approaching a fixed value:

Here is a short table illustrating the point.

Table 1

n	$^{n}1.1$	$^{n}1.2$	$^{n}1.3$	$^{n}1.4$	$^{n}1.5$
1	1.1	1.2	1.3	1.4	1.5
2	1.1105	1.2446	1.4065	1.6017	1.8371
3	1.1116	1.2547	1.4463	1.7142	2.1062
4	1.1118	1.2570	1.4615	1.7803	2.3490
5	1.1118	1.2576	1.4673	1.8203	2.5920
6	1.1118	1.2577	1.4696	1.8450	2.8604
7	1.1118	1.2577	1.4704	1.8604	3.1893
10	1.1118	1.2577	1.4710	1.8800	5.9119
100	1.1118	1.2577	1.4710	1.8867	$>10^{100}$

At some point between x=1.4 and x=1.5, ^{n}x as n approaches infinity transitions from a finite to an infinite value. For brevity, the value of ^{n}x as n approaches infinity will be notated as $^{\infty}x$.

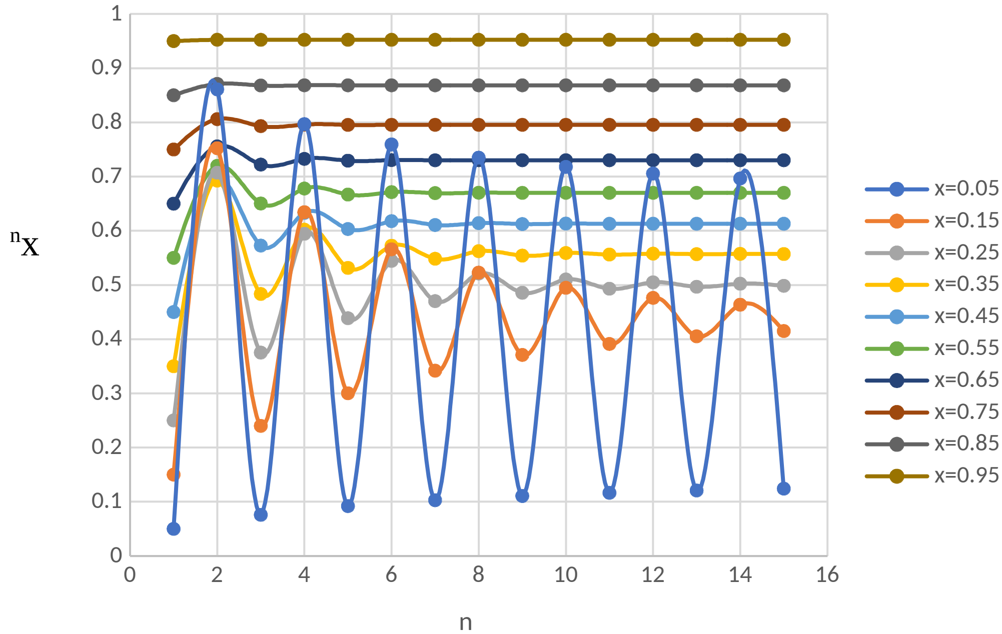
Based on the definition of tetration: $x^{^{\infty}x} = {^{\infty}x}$,

therefore: $x = {^{\infty}x}^{(1/{^{\infty}x})}$.

If we substitute t for $^{\infty}x$: $x = t^{(1/t)}$.

The expression $t^{(1/t)}$, as a function of t, has a maxima at $t = e^{(1/e)}$, approximately 1.44468. That is the value when $^{\infty}x$ transitions from finite to infinite.

When x is between 0 and 1, the behavior of ^{n}x is quite different, as shown in Figure 2.



(Note the dots "•" are calculated values, the line is just an interpolation.)

Figure 2

For x between 0 and 1, ${}^{n}x$ oscillates between an upper and a lower value. For x close to 1, there is little oscillation and it converges rapidly as n goes to ∞. As x gets smaller, the convergence takes longer.

The convergence is very slow for values of x near zero. In fact, at some transition point between 0.05 and 0.10, it does not fully converge, as can be seen in the above Figure.

The transition point can be derived as follows. For small values of x, let $x^{L} = U$, and $x^{U}=L$, where L is the lower value and U the upper value. That means that $U^{(1/L)} = L^{(1/U)}$, and that $L^{L} = U^{U}$. Since L does not equal U, we can assign values to L and U such that L < 1/e and U > 1/e.

Earlier we showed that $x = {}^{\infty}x^{(1/{}^{\infty}x)}$. If we have x and want to solve for ${}^{\infty}x$: we can note that:

If $x = {}^{\infty}x^{(1/{}^{\infty}x)}$, then: $x = 1/[{}^{2}(1/{}^{\infty}x)]$,

therefore: $1/x = {}^{2}(1/{}^{\infty}x)$,

and: $\overline{1/x}\diagdown = (1/{}^{\infty}x)$

and: $1 / \left(\overline{1/x}\diagdown \right) = {}^{\infty}x$

So if ${}^{\infty}x = L$, $1 / \left(\overline{1/x}\diagdown \right) = L < 1/e$

and : $\overline{1/x}\diagdown > e$

Therefore : $1/x > e^{e}$

and : $x < 1/e^e$ (approximately 0.065988)

So $1/e^e$ is the transition point; if $x > 1/e^e$, it converges but if $x < 1/e^e$, ${}^n x$ does not converge.

Steps to Determining Tetration for Non-Integer Heights

Many papers have tried to determine tetration for non-integer heights using the Taylor series. One of the basic assumptions of the Taylor series is infinite differentiability.

However, for $y = {}^x e$, the derivative would be:

$$dy/dx = {}^x e\ {}^{x-1}e\ {}^{x-2}e\ {}^{x-3}e\ldots.. = \prod_{n=0}^{\infty} {}^{(x-n)}e.$$

This would result in an infinite series of complex numbers, even infinite values when x is an integer. This is not calculatable practically. Thus the assumption of infinite differentiability seems non-viable. The possibility of the derivative being infinite at integer values implies that tetration may have multiple solutions at integer values, a condition that would invalidate the use of Taylor series.

The assumption that tetration (or its inverses) could be represented by Taylor series (as in References 1, 2, and 3) seems faulty. Reference 2 is based on Taylor series expansions in Reference 4.

Reference 5 attempts to calculate the derivative of tetration, but step 22 of that derivation is incorrect. The derivative (dy/dx) of a function based on the summation of derivatives of x individual functions is an incorrect process. If this process were used to take the derivative of x^2 (x + x + x + x …. for a total of x times), the result would be x, not 2x.

Review of Inverse Operations and Development of Hybrid Inverse Operation for Exponentiation

For exponentiation, what makes the evaluation with non-integers simple is that if $x^a x^b = x$, then $a + b = 1$. Thus if a and x^a is known, b and x^b can easily be calculated. For each value of $a < 1$, there is a $b < 1$ not equal to a. This because of the commutability of addition, that $a + b = b + a$, therefore if $a + b = c$, $c – b = a$ and $c – a = b$.

The same property holds for multiplication. Since $ab = ba$, if $c = ab$, then $c/a = b$ and $c/b = a$. This characteristic, where a and b can be interchanged and the statement still holds true, could be called reciprocity.

But if we look at exponentiation, this property does not hold, either for roots of logarithms. If $a^b = c$ and $b \neq c$, $\sqrt[b]{c} = a$, but generally $\sqrt[a]{c} \neq b$. Similarly for logarithms, although $\log_a c = b$, $\log_b c$ does not necessarily equal a. However, a hybrid formula combining roots and logarithms that provides reciprocity has been developed:

If $\overline{c^{(1/a)\log_a c}} = b$, then $\overline{c^{(1/b)\log_b c}} = a$.

The reciprocality of this equation can be proven as follows:

If $\overline{c^{(1/a)\log_a c}} = b$, then $c^{(1/a)\log_a c} = b^b$.

Taking the log base c of both sides: $(1/a)\log_a c = b \log_c b$.

Substituting $(1/ \log_c a)$, for $\log_a c$: $1/(a \log_c a) = b \log_c b$.

Inverting both sides: $[a \log_c a] = 1/(b \log_c b)$

Substituting $1/(\log_b c)$ for $\log_c b$: $[a \log_c a] = (\log_b c)/b$

Taking c to the power of both sides: $a^a = c^{(1/b)\log_b c}$

And therefore: $a = \overline{c^{(1/b)\log_b c}}$

This creates a way of calculating ${}^a c$ when ${}^b c$ is known and a and b are between 0 and 1. However, this does not indicate the relationship between a and b, although if a (or b) were close to 1, b (or a) would expected to be close to 0.

Examination of Equations That Could Apply to Multiple Hyperoperations

In order to determine the relationship between a and b, we need to look at examples where an equation for one hyperoperation would apply to the next level hyperoperation.

Not all equations have this property, but many do. The author will present four examples of equations that do have this property.

The first example is if $zf + (1\text{-}f)z = z$. Stepping up one hyperoperation, $z^f z^{(1\text{-}f)} = z$.

The second example is an iterative equation used to calculate square roots, which when stepped up, can be used to calculated the 2nd t-root.

One such formula is $x_2 = 0.5\ (x_1 + A/x_1)$.

Stepping up one hyperoperation, $x_2 = \sqrt{x_1\, A^{(1/x_1)}}$

If A = 1.5 and x_1 = 1.2 (first guess) in both cases, then after 20 iterations, the results would be approximately 1.22474 for the square root and 1.35025 for the 2^{nd} t-root. This means that $1.22474^2 \approx 1.5$ and ${}^{2}1.35025 = 1.35025^{1.35025} \approx 1.5$

The third example is long division (Figure 3). Here is an example for 10/3 (left) and $\log_3 10$ (right) :

```
     3.3333
3 | 10
     9
     10  (multiply by 10)
      9
     ____
      10
       9
     ____
       10
        9
     ____
        10
         9
     ____
          1
```

```
     2.0959
3 | 10
     9
     1.11     division
     1.11^10 = 2.8679 (take to the 10th power)
     1 = 3^0
     2.8679  division
     2.8679^10 = 37648.6 (take to the 10th power)
     19683. = 3^9
     1.9127
     1.9127^10 = 655.3
     243 = 3^5
      2.6967
      2.6967^10 = 20339.4
      19683 = 3^9
      1.033
```

Figure 3

The fourth example is an equation used to calculate logarithms. Calculating $\log_b A$:

$$x_2 = x_1 + A/b^{x_1} - 1$$

To calculate the t-log, we would add one hyperoperation to each step:

$$x_2 = x_1 \, (A^{(1/{}^{x_1}b)}) \, / \, (A^{(1/A)}).$$

We want x_2 to equal x_1 when for formula has converged, so for logarithms, we subtract 1, but for the t-log, we need to divide by $A^{(1/A)}$ for this to occur.

We can use these formulas to calculate $\log_b A$ and $\text{tlog}_b A$. For this example we will use A = 5.9119, b = 1.5, and x_1 = 6. Although we cannot calculate ${}^{x_1}b$ for non-integer values of x_1, the value of ${}^{x}1.5$ increases slowly enough (see Table 1) that linear interpolation is acceptable.

After 100 iterations, the results are $\log_b A$ = 4.38254, and $\text{tlog}_b A$ = 9.99999. (The value of $\text{tlog}_b A$ was intended to be 10 since the tlog is only known for an integer result.)

Relationship Between Heights for Values Between 0 and 1.

Earlier we created a way of calculating ${}^{a}c$ when ${}^{b}c$ is known and a and b are between 0 and 1. But we could not determine the relationship between a and b.

For exponentation, if $c^a c^b = c$, we know that $a + b = 1$. From this can determine that $1 - a = b$ and $1 - b = a$. We could also say that $(½) - a = b - (½)$. Or we can say that $(a - (½))^n = (b - (½))^n$ for even integer values of n.

If we add one hyperoperation, ${}^n(a/\sqrt{1}) = {}^n(b/\sqrt{1})$. Or, to simplify, ${}^n a = {}^n b$. For $a \neq b$, a and b must both be less than 1.

As shown in Figure 4, with exponentiation, for which $(a - (½))^n = (b - (½))^n$; if a is slightly greater than ½ and b is slightly less than ½, as both a and b move towards ½, c^a and c^b move closer together. At the point of equality, $c^a = c^b = \sqrt{c}$.

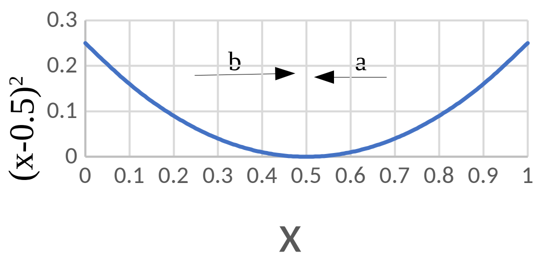


Figure 4

The same thing would happen for tetration, as shown in Figure 5 for which ${}^2a = {}^2b$; if a is slightly greater than 1/e and b is slightly less than 1/e, as both a and b move towards 1/e, ${}^a c$ and ${}^b c$ move closer together. At the point of equality, ${}^a c = {}^b c$ $= \overline{c}\backslash_{e}$.

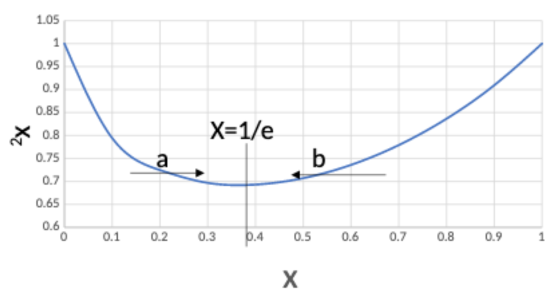


Figure 5

The same are argument would apply for $^{4}a = {}^{4}b$, except that instead of 1/e, a and b would approach an abscissa of the minimum of $y = {}^{4}x$, which is about 0.27469, so $^{0.27469}x = \overline{x}\diagdown^{4}$. The same would apply for 6th, 8th, 10th, 12th, etc. t-roots.

Combination of Non-Integer Heights with Hybrid Inverse Exponentiation Formula

For tetration heights between 0 and 1/e, each even t-root would have a corresponding value for a tetration height between 1/e and 1. For 0.27469, the corresponding tetration height would be 0.46972 since $^{2}0.27469 = {}^{2}0.46972$.

If x=2, $^{0.27469}x = \overline{x}\diagdown^{4} \approx 1.4466$, then $^{0.46972}x = \overline{(^{1.4466}\sqrt{2})^{\log_{1.4466}2}}\diagdown \approx 1.69834$.

Based on these suppositions, we could put together a table for $^{x}2$ for x between 0 and 1. Before we do this, two additional concepts need to be discussed:

First what is $\overline{x}\diagdown^{n}$ as n goes to infinity? (This will be notated as $\overline{x}\diagdown^{\infty}$ for brevity.)

$$\text{If } \overline{x}\diagdown^{\infty} = Y, \text{ then } {}^{\infty}Y = x$$

$$\text{Therefore } Y^{Y^{\infty}} = x,\ Y^{x} = x, \text{ and } Y = {}^{x}\sqrt{x}.$$

Therefore the “infinite” t-root of 2 would be $\sqrt{2}$. As shown previously, $^{\infty}x$ does not converge below $1/e^{e}$. Therefore at $x = 1/^{e}e$, $^{\infty}x$ is at its minimum, and $^{(1/^{e}e)}2 = \sqrt{2}$.

The second concept is that of an inverse function applied to itself; that is f(a,a). For subtraction, this is 0. For division, this is 1. For exponentiation, the result would either be 1 (for logarithms) or $^{a}\sqrt{a}$ for roots. For our hybrid formula, the result would be $\overline{^{a}\sqrt{a}}\diagdown$. Note that this implies that $^{x}2$ is not unique for integer values of x.

So, putting all this into a table for $^{x}2$:

Table 2

x (abscissa of the min of $^nx=y$)	n	nth t-root of 2 = $(2.\sqrt[n]{\ })$	Other solution for ^2x	$^x2$	t-root of this equals
0		1.00	1.00	2.46973	$2^{1.30435}$
0		1.30435	1.00	2.00	$= {}^12$
0.06599	∞	1.41421	0.79889	1.75060	$= (\sqrt[1..41421]{2})^{\log_{1.41421}2}$
0.12786	20	1.41426	0.67880	1.75051	$= (\sqrt[1.41426]{2})^{\log_{1.41426}2}$
0.13357	18	1.41430	0.66904	1.75043	$= (\sqrt[1.4143]{2})^{\log_{1.4143}2}$
0.14049	16	1.41441	0.65745	1.75025	$= (\sqrt[1.41441]{2})^{\log_{1.41441}2}$
0.14928	14	1.41463	0.64306	1.74987	$= (\sqrt[1.41463]{2})^{\log_{1.41463}2}$
0.16039	12	1.41511	0.62541	1.74904	$= (\sqrt[1.41511]{2})^{\log_{1.41511}2}$
0.17507	10	1.41619	0.60290	1.74720	$= (\sqrt[1.41619]{2})^{\log_{1.41619}2}$
0.19548	8	1.41873	0.57301	1.74287	$= (\sqrt[1.41873]{2})^{\log_{1.41873}2}$
0.22565	6	1.42538	0.53149	1.73176	$= (\sqrt[1.42538]{2})^{\log_{1.42538}2}$
0.27469	4	1.44660	0.46973	1.69834	$= (\sqrt[1.4466]{2})^{\log_{1.4466}2}$
0.36788	2	1.55961			

The first column of this table is the x value in $^x2$. Each value in that first column is the abscissa of the minimum value for ^nx, where n is in the second column. The third column is the nth t-root of 2, which would be the value of $^x2$, based on the x in the first column. The first three columns address $^x2$ for x values of 0 at the top, going to $1/^ee$, and then to 1/e at the bottom. The 4th through 6th columns of the table start with the lowest x value at the bottom and work upward. The fourth column is based on the alternate value for ^2x, with x in the first column; essentially ^{2}(1st column) = ^{2}(4th column). The 5th column was calculated with the hybrid inverse exponential function, the expression and the values being in the 6th column, except for the top 2 values, which are based on $2^{(3rd\ column)}$.

Note that, as stated previously, $^x2$ does not have a unique value for integer values of x, essentially result in a vertical slope for $^x2$ at integral values.

Based on the Table 2 values, a plot can be generated (Figure 6):

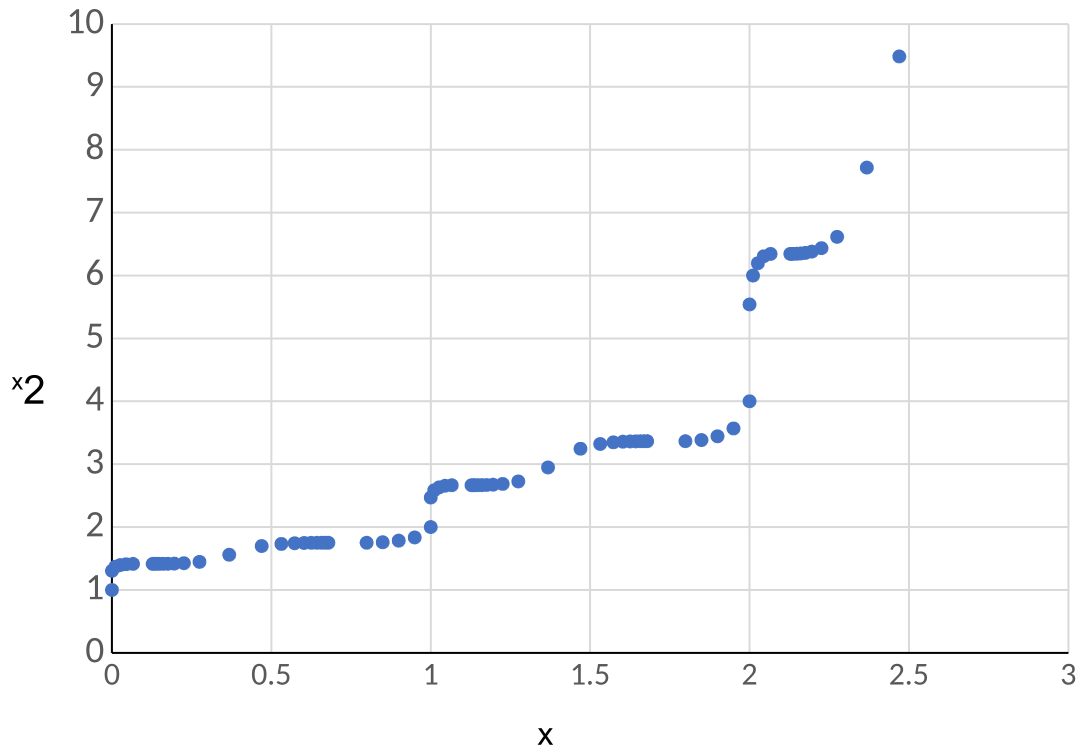


Figure 6

Acknowledgement

The author would like to acknowledge the help of Mark Mortenson, who helped me understand the referenced material on tetration, as well as writing my paper in a way that was understandable to other mathematicians.

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